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THIN POSITION AND BRIDGE NUMBER FOR KNOTS IN THE 3-SPHERE

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The bridge number of a knot in S^3, introduced by Schubert [1] is a classical and well-understood knot invariant. The concept of *thin position* for a knot was developed fairly recently by Gabai [2]. It has proved to be an extraordinarily useful notion, playing a key role in Gabai's proof of property R as well as Gordon–Luecke's solution of the knot complement problem [3]. The purpose of this paper is to examine the relation between bridge number and thin position. We show that either a knot in thin position is also in the position which realizes its bridge number or the knot has an incompressible meridianal planar surface properly imbedded in its complement. The second possibility implies that the knot has a generalized tangle decomposition along an incompressible punctured 2-sphere. Using results from [4, 5], we note that if thin position for a knot K is *not* bridge position, then there exists a closed incompressible surface in the complement of the knot (Corollary 3) and that the tunnel number of the knot is strictly greater than one (Corollary 4).

We need a few definitions prior to starting the main theorem.

Definitions: Let K be a knot in the 3-sphere. The *complement of* K is S^3 with a small open neighborhood of K, $n(K)$, removed. A *meridianal* planar surface in the complement of K is a planar surface properly imbedded in the knot complement with meridianal boundary components. A boundary parallel annulus with meridianal boundary components is a *trivial* meridianal planar surface. We recall the definition of *thin position*: let $h{:}\,[S^3$-(two points)$] \to [0, 1]$ be a height function on S^3 that restricts to a Morse function on K. Choose a regular value t_i between each pair of adjacent critical values of $h|_K$. Define the *width of K with respect to h* to be the sum over i of [the number of times K intersects $h^{-1}(t_i)$]. Define the *width of* K to be the minimum width of K with respect to h over all h. Suppose the function h realizes the width of K. We can assume by an isotopy that the foliating 2-spheres of S^3-(two points) for h are round, and we call the image of K under this isotopy *thin position* for K. For any regular value t_0 of $h|_K$ we can find a neighborhood $n(K)$ such that $P = h^{-1}(t_0) - n(K)$ is a meridianal planar surface in the complement of K. A *strict upper disk* for P is a disk D such that $\partial D = \alpha \cup \beta$, where α is an arc properly imbedded in P, β is an arc imbedded on the boundary of $n(K)$, parallel to an arc of K, $\partial\alpha = \partial\beta$, int$(D)$ intersects P in simple closed curves, and a small product neighborhood of α in D lies on the side of P containing $h^{-1}(1)$, i.e., it lies *above* P. Strict lower disks are defined similarly. S is a *thin 2-sphere* for K with respect to the height function h if $S = h^{-1}(t_0)$ for some regular value

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t_0 where t_0 lies between adjacent critical values x and y of h, where x is a minimum of K lying above t_0 and y is a maximum of K lying below t_0.

Our main theorem is:

THEOREM 1. *If the complement of a knot K in S^3 contains no non-trivial incompressible meridianal planar surface, then a height function h realizing the width of K also realizes the bridge number of K.*

The proof of Theorem 1 requires Lemma 2.

LEMMA 2. *If K in S^3 is in thin position with respect to the height function h and S is a thin 2-sphere for K with respect to h, then there are no strict upper or lower disks for the meridianal planar surface $P = S - n(K)$.*

Proof of Lemma 2. Suppose D is a strict upper disk for P, with $\partial D = \alpha \cup \beta$. Via an isotopy across D, replace the arc of K corresponding to β with the arc α. Since the critical point immediately above P is a minimum of K, and since β must contain at least one maximum of K, this either:

1. pulls at least one maximum of K below one minimum of K without introducing any additional critical points, or
2. eliminates at least two critical points (if β itself contains a minimum)

hence reducing the width of K. This contradiction proves the lemma. □

Proof of Theorem 1. Suppose $K \subset S^3$ is in thin position, with height function h. If there are no thin spheres for K with respect to h, then K is in bridge position and we are done.

Suppose S is a thin 2-sphere for K with respect to h. Let $P = S - n(K)$. Compress P as much as possible in the complement of K. The compressions may take place to either side of P, and may have to be done in several steps. Let $\tilde{P}$ be the resulting collection of meridianal planar surfaces. $\tilde{P}$ is incompressible in the complement of K, hence each component of $\tilde{P}$ is incompressible in the complement of K. Let P' be an innermost component of $\tilde{P}$, i.e., $S' = P' \cup$ (meridian disks of K) bounds a 3-ball with interior disjoint from $\tilde{P}$. There are two possibilities:

1. P' is a boundary parallel annulus.
2. P' is a non-trivial incompressible meridianal planar surface in the complement of K.

If option 2 occurs, we are done.

Suppose P' is a boundary parallel annulus. Let D be a disk such that ∂D consists of two arcs α and β, $\partial\alpha = \partial\beta$, with α a properly imbedded essential arc in P', β an arc imbedded on the boundary of a neighborhood of K, and $\mathrm{int}(D)$ disjoint from $\tilde{P}$. Now reverse the compressions on $\tilde{P}$ to reassemble P. This means that the components of $\tilde{P}$ are reattached by tubes, which may run through each other, and which may intersect D. However, since the tubes can be chosen to be disjoint from a neighborhood of ∂D, D persists as a strict upper or lower disk for S, contradicting Lemma 2. Hence option 1 cannot occur. □

COROLLARY 3. *Let K be a knot in S^3 such that thin position for K is not bridge position. Then there exists a closed incompressible surface in the complement of K.*

Proof of Corollary 3. By Theorem 1, there exists a non-trivial incompressible meridianal planar surface properly imbedded in the complement of K. By [4, Theorem 2.0.3], this implies that there exists a closed incompressible non-boundary parallel surface in the complement of K.

COROLLARY 4. *Let K be a knot in S^3 with tunnel number one. Then thin position for K is bridge position.*

Proof of Corollary 4. This follows from Theorem 1 and Corollary 1.2 of [5]: *A tunnel number one knot in S^3 is n-string prime.*

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